Math 1510 Week 10, 11

Reduction formula

$\int x^n e^{ax}\,dx \quad n \geq 0,\ a \neq 0$ is constant

Goal: Lower degree of x

Try: Integration by parts:

Integrate e^{ax}, differentiate x^n

Sol Let $I_n = \int x^n e^{ax}\,dx$.

$$I_n = \frac{1}{a}\int x^n e^{ax}\,d(ax)$$

$$= \frac{1}{a}\int x^n\,de^{ax}$$

$$= \frac{1}{a}\left(x^n e^{ax} - \int e^{ax}\,dx^n\right)$$

$$= \frac{1}{a}\left(x^n e^{ax} - \int n x^{n-1} e^{ax}\,dx\right)$$

$$I_n = \frac{1}{a}\left(x^n e^{ax} - n I_{n-1}\right)$$ Reduction formula

eg Find $\int x^3 e^{-x}\,dx$.

Sol Take $a=-1$ in the last reduction formula.

Then $I_n = n I_{n-1} - x^n e^{-x}$

$$I_3 = 3I_2 - x^3 e^{-x}$$

$$= 3(2I_1 - x^2 e^{-x}) - x^3 e^{-x}$$

$$= 6I_1 - 3x^2 e^{-x} - x^3 e^{-x}$$

$$= 6(I_0 - xe^{-x}) - 3x^2 e^{-x} - x^3 e^{-x}$$

$$= 6\int e^{-x}\,dx - 6xe^{-x} - 3x^2 e^{-x} - x^3 e^{-x}$$

$$= -e^{-x}(6 + 6x + 3x^2 + x^3) + C$$

Ex Find a Reduction formla for $I_n = \int (\ln x)^n\,dx$

Ans $I_n = x(\ln x)^n - n I_{n-1}$

eg Let $I_n = \int x^n \cos x \, dx$ for $n \geq 0$.

Find a reduction formula for I_n.

Sol For $n \geq 2$

$$I_n = \int x^n \cos x \, dx$$

(To lower n, should integrate $\cos x$, differentiate x^n)

$$= \int x^n \, d\sin x$$

$$= x^n \sin x - \int \sin x \, dx^n$$

$$= x^n \sin x - n \int x^{n-1} \sin x \, dx$$

Another integration by parts

$$= x^n \sin x + n \int x^{n-1} \, d\cos x$$

$$= x^n \sin x + n\left(x^{n-1}\cos x - \int \cos x \, dx^{n-1}\right)$$

$$= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx$$

$$= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$$

eg Let $I_n = \int \frac{1}{x^n(1+x)} dx$ for $n = 0, 1, 2 \cdots$

Find a reduction formula for I_n.

(Hint: Write $1 = (1+x) - x$)

Sol

$$I_n = \int \frac{(1+x) - x}{x^n(1+x)} dx$$

$$= \int \left[\frac{1}{x^n} - \frac{1}{x^{n-1}(1+x)}\right] dx$$

Be careful

$$\overset{*}{=} \begin{cases} \frac{1}{(1-n)x^{n-1}} - I_{n-1} & \text{if } n \geq 2 \\ \ln|x| - I_0 & \text{if } n = 1 \end{cases}$$

eg $I_0 = \ln|1+x| + C$

$I_1 = \ln|x| - \ln|1+x| + C$

$I_2 = \frac{-1}{x} - \ln|x| + \ln|1+x| + C$

eg Find Reduction formula for

$$I_n = \int \tan^n x \, dx \quad \text{and} \quad J_n = \int \sec^n x \, dx$$

Sol

$$I_n = \int \tan^n x \, dx \quad , n \geq 2$$

$$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \int \tan^{n-2} x \, d\tan x - I_{n-2}$$

$$= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$$

$$J_n = \int \sec^{n-2} x \sec^2 x \, dx$$

$$= \int \sec^{n-2} x \, d\tan x$$

$$= \sec^{n-2} x \tan x - \int \tan x \, d\sec^{n-2} x$$

$$= \sec^{n-2} x \tan x - (n-2) \int \tan x \sec^{n-3} x \, (\sec x \tan x) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2)(J_n - J_{n-2})$$

$$\therefore (n-1) J_n = \sec^{n-2} x \tan x + (n-2) J_{n-2}$$

$$J_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} J_{n-2}$$

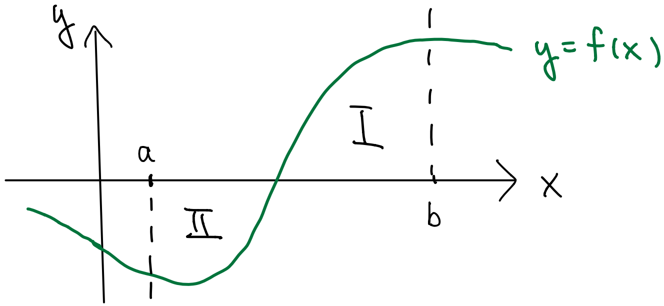
Definite Integral

Defn For $a \le b$, define

$$\int_a^b f(x)\,dx = \text{signed area under the graph of } f(x) \text{ between } x=a \text{ and } x=b$$

If $a > b$, define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Picture $(a<b)$



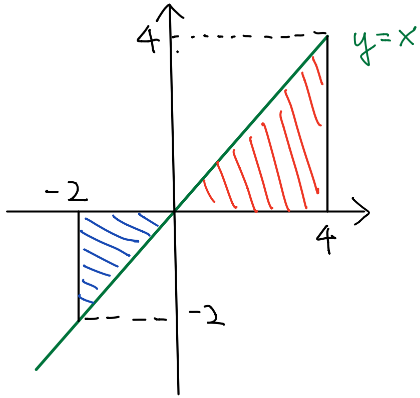
$$\int_a^b f(x)\,dx = \text{Area of I} - \text{Area of II}$$

Rmk $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

The name of the variable is not important

It is called dummy variable

eg $f(x) = x$



$$\int_{-2}^4 x\,dx = \text{Area of } \text{(red triangle)} - \text{Area of } \text{(blue triangle)}$$

$$= \frac{1}{2}(4)(4) - \frac{1}{2}(2)(2)$$

$$= 6$$

Riemann Sum

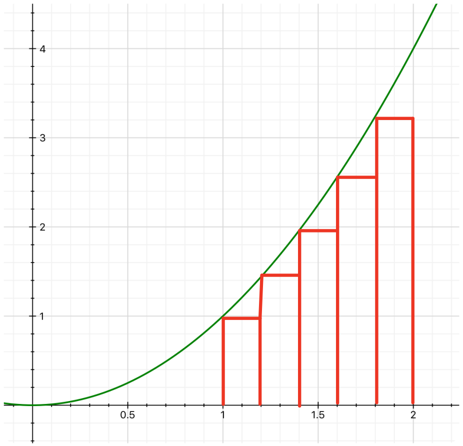
eg Let $f(x)=x^2$. Approximate $\int_1^2 f(x)\,dx$:

Divide $[1,2]$ into n intervals $I_1, I_2, \cdots I_n$ $\Rightarrow$ Length of each $I_k = \frac{2-1}{n} = \frac{1}{n}$

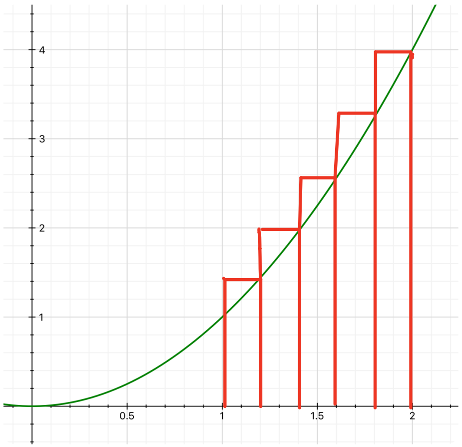
$I_1 = [1, 1+\frac{1}{n}]$, $I_2 = [1+\frac{1}{n}, 1+\frac{2}{n}]$ $\cdots$ $I_k = [1+\frac{k-1}{n}, 1+\frac{k}{n}]$ $\cdots$

Approximate Area under graph using rectangles (Riemann Sum)

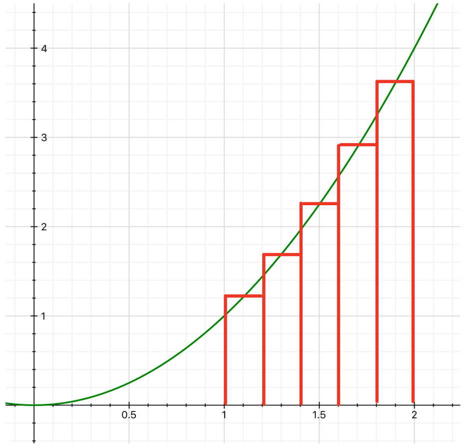
Picture for $n=5$ (3 different ways below)



"Left" Riemann sum $= \sum_{k=1}^{n} \frac{1}{n} f\left(1+\frac{k-1}{n}\right)$



"Right" Riemann sum $= \sum_{k=1}^{n} \frac{1}{n} f\left(1+\frac{k}{n}\right)$



"Mid-point" Riemann sum $= \sum_{k=1}^{n} \frac{1}{n} f\left(1+\frac{2k-1}{2n}\right)$

We approximate $\int_1^2 f(x)\,dx = \int_1^2 x^2\,dx$

using "Right Rieman Sum"

Total area of rectangles

$$= \sum_{k=1}^{n} \frac{1}{n} f\left(1+\frac{k}{n}\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n} \left(1+\frac{k}{n}\right)^2$$

$$= \sum_{k=1}^{n} \frac{1}{n} \left(1+\frac{2k}{n}+\frac{k^2}{n^2}\right)$$

$$= \frac{1}{n}\left(\sum_{k=1}^{n} 1\right) + \frac{2}{n^2}\left(\sum_{k=1}^{n} k\right) + \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2\right)$$

$$= \frac{1}{n}\cdot n + \frac{2}{n^2}\cdot\frac{1}{2}(n)(n+1) + \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$$

$$= 1 + \left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Formula

$$\sum_{k=m}^{n} af(k) + bg(k) = a\sum_{k=m}^{n} f(k) + b\sum_{k=m}^{n} g(k)$$

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(n+2)$$

Bigger $n \Rightarrow$ Better approximation

Take $n \to \infty$

Total area of rectangles $\longrightarrow$ Area under graph

$$\int_1^2 x^2\,dx = \lim_{n\to\infty}\left[1 + \left(1+\frac{1}{n}\right) + \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 1 + (1+0) + \frac{1}{6}(1+0)(2+0)$$

$$= \frac{7}{3}$$

Properties of Definite Integrals

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

④ For constants α, β,

$$\int_a^b \left(\alpha f(x) + \beta g(x)\right) dx$$

$$= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

⑤ Let $a \le b$. If $f(x) \le g(x)$ on $[a,b]$,

then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

ⓐ If $m \le f(x) \le M$ on $[a,b]$

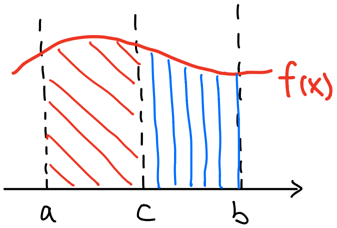
then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

ⓑ Note $-|f(x)| \le f(x) \le |f(x)|$ on $[a,b]$

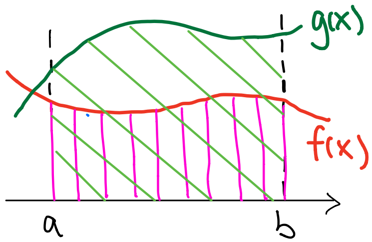
$$\Rightarrow -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

i.e $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

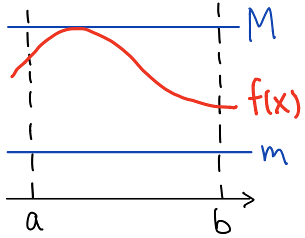
③



⑤



5a



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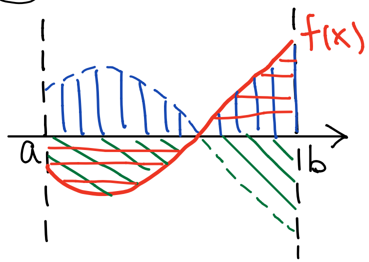
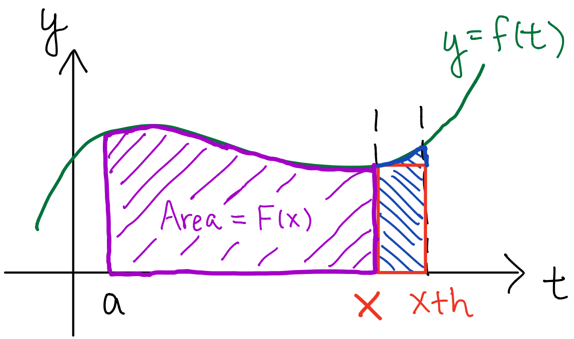


Fundamental Theorem of Calculus

Given a function f, let

$$F(x) = \int_a^x f(t)\,dt$$

$=$ Signed area under the graph of f from a to x



$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt \quad \text{(Area of)}$$

$$\approx f(x)\cdot h \quad \text{(Area of)}$$

Fundamental Theorem of Calculus (FTC)

① (Differentiate an integral)

Let $f(t)$ be a continuous function and

$$F(x) = \int_a^x f(t)\,dt$$

Then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$\therefore$ F is an anti-derivative of $f(x)$

② (Integrate a derivative)

If $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Rmk FTC can be proved using a MVT for integration

Differentiate an integral

① of FTC:

$$\boxed{f \text{ is continuous} \Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

eg Find $g'(x)$ if

a. $g(x) = \int_1^x \ln(t^2+4)\,dt$

Sol FTC $\Rightarrow$ $g'(x) = \ln(x^2+4)$

b. $g(x) = \int_{-1}^{x^2} e^{\sin t}\,dt$

Sol $g'(x) = \left(\frac{d}{dx^2}\int_{-1}^{x^2} e^{\sin t}\,dt\right)\frac{dx^2}{dx}$

$$= e^{\sin x^2} \cdot 2x$$

c. $g(x) = \int_{x^4}^{1} \sec t\,dt$

Sol $g(x) = -\int_1^{x^4} \sec t\,dt$

$$g'(x) = -\left(\frac{d}{dx^4}\int_1^{x^4} \sec t\,dt\right)\left(\frac{dx^4}{dx}\right)$$

$$= -4x^3 \sec x^4$$

d. $g(x) = \int_{\sin x}^{\cos x} |t|\,dt$

Sol $g(x) = \int_0^{\cos x} |t|\,dt + \int_{\sin x}^{0} |t|\,dt$

$$= \int_0^{\cos x} |t|\,dt - \int_0^{\sin x} |t|\,dt$$

$$g'(x) = |\cos x|(\cos x)' - |\sin x|(\sin x)'$$

$$= -|\cos x|\sin x - |\sin x|\cos x$$

Compute definite integral

② of FTC:

> If $\int f(x)\,dx = F(x)$
>
> Then $\int_a^b f(x)\,dx = F(b) - F(a)$

Notation $[F(x)]_a^b = F(x)\Big|_a^b = F(b) - F(a)$

eg $\int_0^2 (3x^2+1)\,dx = [x^3+x]_0^2$

$$= (2^3+2) - (0^3+0)$$

$$= 10$$

Q Can we use another derivative?

A Sure!

$$\int_0^2 (3x^2+1)\,dx = [\overbrace{x^3+x+2}^{\text{Another derivative}}]_0^2$$

$$= (2^3+2+2) - (0^3+0+2)$$

$$= 10$$

Cancel

Definite integral by Substitution

Let $u = u(x)$

$$\boxed{\underbrace{\int_a^b f(u(x))\,u'(x)\,dx}_{\text{in terms of } x} = \underbrace{\int_{u(a)}^{u(b)} f(u)\,du}_{\text{in terms of } u}}$$

eg $\int_0^2 x e^{x^2}\,dx$

Sol Let $u = x^2$ $du = 2x\,dx$

When $x=2$, $u = 2^2 = 4$

When $x=0$, $u = 0^2 = 0$

$$\int_0^2 x e^{x^2}\,dx = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}[e^u]_0^4 = \frac{1}{2}(e^4 - e^0)$$

$$= \frac{1}{2}(e^4 - 1)$$

Alternative Sol (Easier for simple substitution)

$$\int_0^2 x e^{x^2}\,dx = \frac{1}{2}\int_0^2 e^{x^2}\,dx^2 = \frac{1}{2}[e^{x^2}]_0^2 = \frac{1}{2}(e^4 - 1)$$

eg $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$

Sol Let $t = \tan\frac{x}{2}$

Then $dx = \frac{2dt}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$

When $x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2+2t}$$

$$= \int_0^1 \frac{2\,d(1+t)}{(1+t)^2}$$

$$= \left[-\frac{2}{1+t}\right]_0^1$$

$$= -2\left(\frac{1}{2}-1\right) = 1$$

Integration by parts for definite integral

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

eg $\int_1^n \ln x\,dx$

$$= [x\ln x]_1^n - \int_1^n x\,d\ln x$$

$$= [n\ln n - (1)\ln 1] - \int_1^n x \cdot \frac{1}{x}\,dx$$

$$= n\ln n - [x]_1^n$$

$$= n\ln n - (n-1)$$

$$= n\ln n - n + 1$$

Reduction formula for definite integral

eg Let $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$

① Find Reduction formula ② Find I_5.

Sol

$$I_n = \int_0^1 x^{n-1}\sqrt{1-x^2}\, x\,dx$$

$$= -\frac{1}{2}\int_0^1 x^{n-1}(1-x^2)^{\frac{1}{2}}\, d(1-x^2)$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^1 x^{n-1}\, d(1-x^2)^{\frac{3}{2}}$$

$$= -\frac{1}{3}\left(\left[x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 - \int_0^1 (1-x^2)^{\frac{3}{2}}\, dx^{n-1}\right)$$

$$= -\frac{1}{3}\left[0-(n-1)\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\, dx\right]$$

$$= \frac{n-1}{3}\left(\int_0^1 x^{n-2}\sqrt{1-x^2}\, dx - \int_0^1 x^n\sqrt{1-x^2}\, dx\right)$$

$$= \frac{n-1}{3}\left(I_{n-2} - I_n\right)$$

Hence,

$$\left(1+\frac{n-1}{3}\right) I_n = \frac{n-1}{3} I_{n-2}$$

$$(3+n-1) I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n+2} I_{n-2}$$

$$\therefore I_5 = \frac{4}{7} I_3$$

$$= \frac{4}{7}\cdot\frac{2}{5} I_1$$

$$= \frac{8}{35}\int_0^1 x\sqrt{1-x^2}\, dx$$

$$= \frac{8}{35}\left(-\frac{1}{2}\right)\int_0^1 \sqrt{1-x^2}\, d(1-x^2)$$

$$= -\frac{4}{35}\cdot\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$$

$$= -\frac{4}{35}\left(0-\frac{2}{3}\right) = \frac{8}{105}$$

Improper Integral

Sometimes it is possible to integrate a function over an interval of infinite length

Def If $f(x)$ is defined on $[a,\infty)$ and $\lim_{t\to\infty}\int_a^t f(x)\,dx$ exists, then define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

Similarly, if $f(x)$ is defined on $(-\infty,b]$ and $\lim_{t\to-\infty}\int_t^b f(x)\,dx$ exists, then define

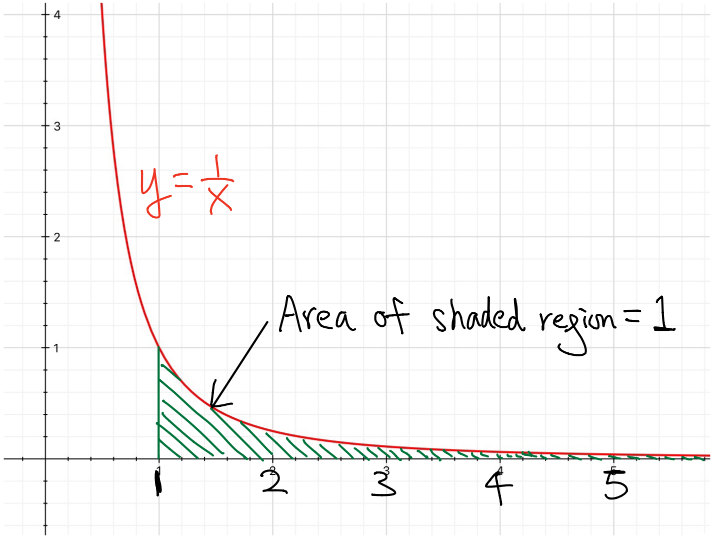
$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

eg. $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right)$$

$$= 1$$

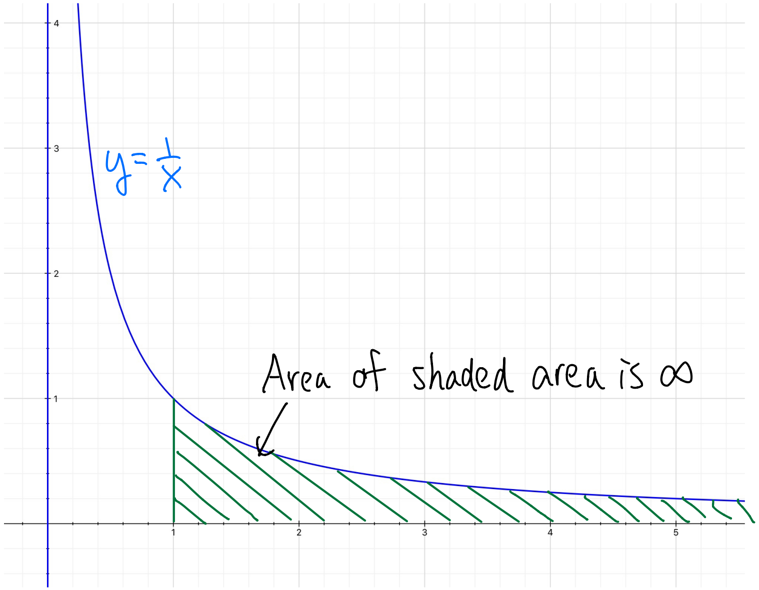


$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

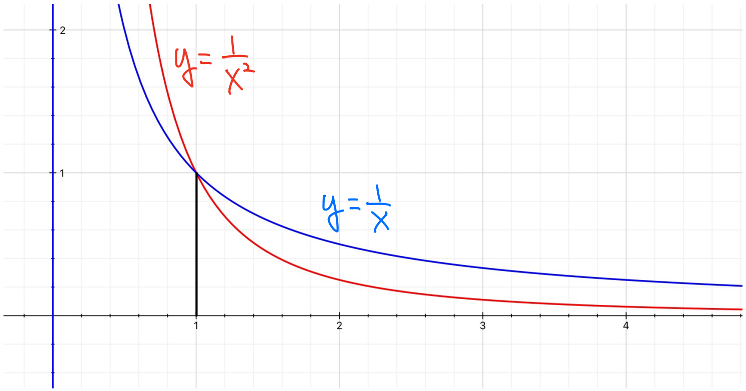
$$= \lim_{t\to\infty}\Big[\ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} \ln|t|$$

$$= \infty \quad (\text{DNE})$$



Compare the two cases



Over the interval $[1,\infty)$,

Area under $y=\frac{1}{x^2}$ $\ll$ Area under $y=\frac{1}{x}$

It may also be possible to integrate a function not defined at an end point of the interval of integration.

<u>Def</u> If $f(x)$ is defined on $[a,b)$ and $\lim_{t\to b^-}\int_a^t f(x)\,dx$ exists, then define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

Similarly, if $f(x)$ is defined on $(a,b]$ and $\lim_{t\to a^+}\int_t^b f(x)\,dx$ exists, then define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

The integrals defined in the last two definitions are called <u>improper integrals</u>

An improper integral is called $\begin{cases}\text{convergent}\\ \text{divergent}\end{cases}$ if the associated limit $\begin{cases}\text{exists}\\ \text{DNE}\end{cases}$

<u>eg</u> Note $\tan x$ is not defined at $\frac{\pi}{2}$ with

$$\lim_{x\to\frac{\pi}{2}^-}\tan x = +\infty$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\tan x\,dx &= \lim_{t\to\frac{\pi}{2}^-}\int_0^t \tan x\,dx\\ &= \lim_{t\to\frac{\pi}{2}^-}\Big[\ln|\sec x|\Big]_0^t\\ &= \lim_{t\to\frac{\pi}{2}^-}\ln|\sec t| - \ln|\sec 0|\\ &= \lim_{t\to\frac{\pi}{2}^-}\ln|\sec t|\\ &= \infty \quad (\text{DNE})\end{aligned}$$

$\therefore \int_0^{\frac{\pi}{2}}\tan x\,dx$ is divergent eg $\frac{1}{\sqrt{x}}$ is not defined at 0 with

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t \to 0^+} \left[2\sqrt{x}\right]_t^1$$

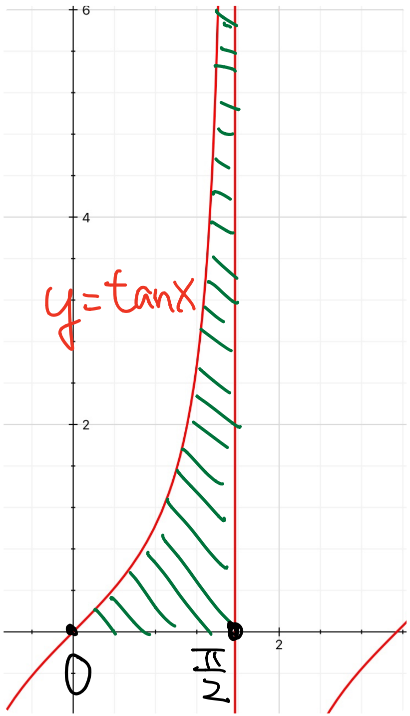
$$= \lim_{t \to 0^+} \left(2 - 2\sqrt{t}\right)$$

$$= 2 - 2\sqrt{0}$$

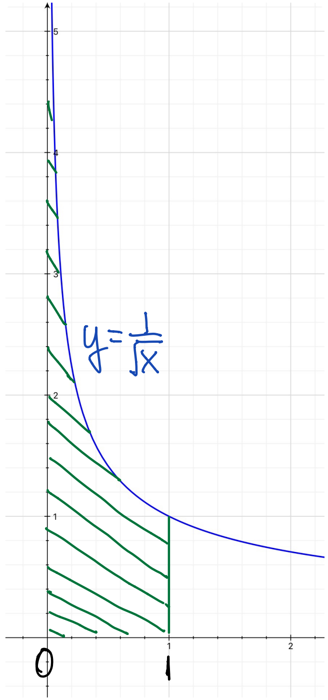
$$= 2$$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent

Graphs



Area $= \infty$



Area $= 2$

The examples of improper integrals discussed involve taking limit at one endpoint.

It is possible for an improper integral to have limits involved at both endpoints

Such an integral can be computed by splitting the integral into two.

e.g. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

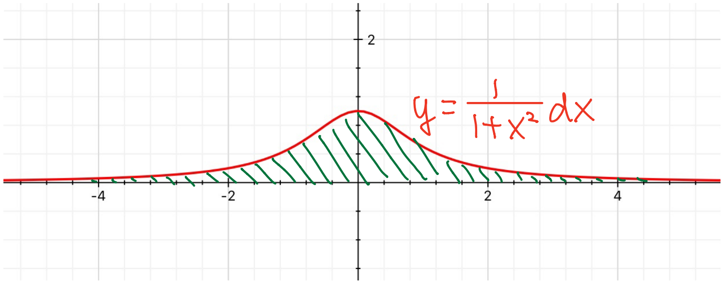
$\displaystyle = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$\displaystyle = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{1+x^2}$

$\displaystyle = \lim_{t\to-\infty} (\arctan 0 - \arctan t) + \lim_{t\to\infty} (\arctan t - \arctan 0)$

$\displaystyle = \left[0 - \left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2} - 0\right]$

$= \pi$



Area of shaded region $= \pi$